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Letter to the Editor

Best Approximation to $1-x^m$ by Special Rational Functions

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In a recent note [1], we have shown that the maximal error in the best uniform approximation to $(1-x)$ by rational functions $P(x)/Q(x)$, where $P(x)$, $Q(x)$ are polynomials of degree $\leqslant n$ having non-negative, non-increasing coefficients, is $(n+2)^{-1}$. Now it is natural to ask, given an integer $m \geqslant 1$, how close can one approximate $(1-x^m)$ on $[0, 1]$ by $P(x)/Q(x)$, where the polynomial $P(x)$ has non-negative, non-decreasing coefficients and degree $\leqslant m-1$, and the polynomial $Q(x)$ has non-negative, non-increasing coefficients and degree $\leqslant n$.

THEOREM 1. *Let m, n be integers, $1 \leqslant m \leqslant n+1$. Then*

$$\left\| (1-x^m) - \frac{(n+1)\sum_{i=0}^{m-1} x^i}{(n+m+1)\sum_{i=0}^{n} x^i} \right\|_{L^\infty_{[0,1]}} = \frac{m}{m+n+1}. \tag{1}$$

THEOREM 2. *Let $P(x)$ be a real polynomial of degree $\leqslant m-1$ $(m \geqslant 1)$ having non-negative, non-decreasing coefficients $a_j \equiv P_j(0)/j!$ and $Q(x)$ a real polynomial of degree $\leqslant n$ $(n \geqslant 0)$ having non-negative, non-increasing coefficients $b_j \equiv Q^{(j)}(0)/j!$, $Q(0) > 0$. Then*

$$\left\| (1-x^m) - \frac{P(x)}{Q(x)} \right\|_{L^\infty_{[0,1]}} \geqslant \frac{m}{m+n+1}. \tag{2}$$

Proof of (1). For $0 \leqslant x \leqslant 1$ satisfying $x^{n+1} \leqslant m(m+n+1)^{-1}$, we have

$$0 \leqslant (1-x^m) - \frac{(n+1)\sum_{j=0}^{m-1} x^j}{(n+m+1)\sum_{j=0}^{n} x^j} = \frac{\left(\sum_{j=0}^{m-1} x^j\right)\left(m/(m+n+1) - x^{n+1}\right)}{\sum_{j=0}^{n} x^j}$$
$$\leqslant m/(m+n+1).$$

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For $0 \leqslant x \leqslant 1$ satisfying $m(n+m+1)^{-1} < x^{n+1}$, we have

$$0 < \frac{(n+1)\sum_{j=0}^{m-1} x^j}{(n+m+1)\sum_{j=0}^{n} x^j} - (1-x^m)$$

$$\leqslant \frac{(\sum_{j=0}^{m-1} x^j)(x^{n+1} - m/(n+m+1))}{\sum_{j=0}^{n} x^j} \leqslant \frac{mx^{n+1} - m^2 x^{n+1}/(m+n+1)}{(n+1)x^{n+1}}$$

$$= \frac{m}{m+n+1}.$$

Equality (1) now follows from (2).

Proof of (2). Set

$$\left\| 1 - x^m - \frac{P(x)}{Q(x)} \right\|_{L^{\infty}_{[0,1]}} = \varepsilon.$$

Then

$$\varepsilon \geqslant \frac{P(1)}{Q(1)} \geqslant \frac{ma_0}{(n+1)b_0} = \frac{m}{n+1}\left(\frac{a_0}{b_0} - 1\right) + \frac{m}{n+1} \geqslant -\frac{m}{n+1}\varepsilon + \frac{m}{n+1}.$$

Hence $\varepsilon \geqslant m/(m+n+1)$.

REFERENCE

1. A. R. REDDY, A note on a result of Bernstein, *J. Approx. Theory* 47 (1986), 336–340.

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